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A Stringy Wave Function for an S^3 Cosmology

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Using the recent observations of the relation between Hartle-Hawking wave function and topological string partition function, we propose a wave function for scalar metric fluctuations on $\mathbf{S}^3$ embedded in a Calabi-Yau. This problem maps to a study of non-critical bosonic string propagating on a circle at the self-dual radius. This can be viewed as a stringy toy model for a quantum cosmology.

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1. Introduction

The notion of the wave function of the universe, in the mini-superspace description a la Hartle-Hawking [1], has recently been made precise in the context of a certain class of string compactifications [2]. In particular, this work provided an explanation for the appearance of a topological string wave function in the conjecture of [3] relating the entropy of certain extremal 4d black holes with the topological string wave function. It is natural to ask if we can extend this picture to obtain a more realistic quantum cosmology within string theory. The aim of the present paper is to take a modest step in this direction.

The basic setup in [2] was flux compactifications of type II string theory on a Calabi-Yau three-fold times $\mathbf{S}^2 \times \mathbf{S}^1$, and providing a wave function on moduli space of Calabi-Yau and the overall size of $\mathbf{S}^2$. In particular for a given choice of flux in the Calabi-Yau, labeled by magnetic and electric fluxes (P^I, Q_I), we have a wave function, $\psi_{P,Q}(\Phi^I)$, depending on (real) moduli of Calabi-Yau. This wave function is peaked at the attractor values of the moduli of the Calabi-Yau. Also, in general, this wave function depends only on the BPS subspace of the field configurations which thus yields a rather limited information about the full Calabi-Yau wave function. One would like to have a wave function which depends on more local data of the Calabi-Yau geometry, rather than just global moduli. This may seem to be in contradiction with the requirement that the data depends only on BPS quantities. However this need not be the case, as we will now explain.

For concreteness, let us take type IIB superstring compactified on a Calabi-Yau three-fold and consider the shape of a particular special Lagrangian 3-cycle L inside the Calabi-Yau. For instance, this may be a natural setup for a toy model of our universe obtained by wrapping some D-branes on L. In this setup, the question about the wave function as a function of the shape of L translates into the wave function for our universe. In general, varying the moduli of Calabi-Yau will induce changes in the shape of L. So, at least we have a wave function on a subset of the moduli of L. More precisely, since Calabi-Yau space has a 3-form which coincides with the volume form on special Lagrangian submanifolds, we are effectively asking about a wave function for some subset of local volume fluctuations on L. On the other hand, since the issue is local, we can consider a local model of Calabi-Yau near L, which is given by T^*L. In this context, global aspects of Calabi-Yau will not provide any obstruction in arbitrary local deformations of the shape of L. We could thus write a wave function which is a function of *arbitrary local volume fluctuations of* L. In particular, if we know how to compute topological string wave function on T^*L we will be

able to write the full wave function for arbitrary local volume fluctuations (scalar metric perturbations) of L.

A particularly interesting choice of L is $L = \mathbf{S}^3$. Not only is this the most natural choice in the context of quantum cosmology, but luckily it also turns out to be the case already well studied in topological string theory: As is well known, the topological B model on the conifold $T^*\mathbf{S}^3$ gets mapped to non-critical bosonic string propagating on a circle of self-dual radius [4]. Hence, we can use the results on the $c = 1$ non-critical bosonic string theory to write a wave function for scalar metric fluctuations of the $\mathbf{S}^3$. This is the main goal of the present paper. We will show how the known results of the non-critical bosonic strings can be used to yield arbitrary 2-point fluctuations. In particular we find the following result

$$\langle \phi_{\vec{k}} \phi_{-\vec{k}} \rangle \sim g_s^2 |\vec{k}| \tag{1.1}$$

where $\phi_{\vec{k}}$ denotes the Fourier modes of the conformal rescalings of the metric, which differs from the scale invariant spectrum in the standard cosmology:

$$\langle \phi_{\vec{k}} \phi_{-\vec{k}} \rangle \sim |\vec{k}|^{-3} \tag{1.2}$$

One can also in principle compute arbitrary n-point fluctuations. However, in general, for this one would need to know arbitrary momentum and winding correlation functions of the non-critical bosonic string which are not yet available (see, however, the recent work [5]). Nevertheless, from the known results about the correlation functions of the momentum modes of $c = 1$ string theory we can obtain arbitrary n-point fluctuations for scalar fluctuations on a large circle $\mathbf{S}^1 \subset \mathbf{S}^3$.

The organization of this paper is as follows: In section 2, we review the notion of the wave function for topological strings and its relation to the wave function for moduli of a Calabi-Yau in flux compactifications [2]. In section 3, we review non-critical bosonic string theory on a circle of self-dual radius and its relation to the topological B model on $T^*\mathbf{S}^3$. In section 4, we use these relations to compute the wave function for local volume fluctuations on $\mathbf{S}^3$ and compute some n-point correlation functions. We also discuss some possible toy model cosmologies based on $\mathbf{S}^3$. Finally, in section 5, we end with the discussion of some open questions and directions for future research.

2. Stringy Hartle-Hawking Wave Function

In this section we briefly review the work of [2]. Consider a flux compactification of type IIB string on a Calabi-Yau space M times $\mathbf{S}^2 \times \mathbf{S}^1$, with a 5-form field strength flux threading through $\mathbf{S}^2$ and a 3-cycle of M. We choose a canonical symplectic basis for the three cycles on M, denoted by A_I, B^J. In this basis, the magnetic/electric flux can be denoted by (P^I, Q_J). The wave function of the "universe" in the mini-superspace will be a function of the moduli of M and the sizes of $\mathbf{S}^2$ and $\mathbf{S}^1$. It turns out that it does not depend on the size[1] of the $\mathbf{S}^1$ and its dependence on the size of $\mathbf{S}^2$ can be recast by writing the wave function in terms of the projective coordinate on the moduli space of M.

The moduli space of a Calabi-Yau is naturally parameterized by the periods of the holomorphic 3-form Ω on the 3-cycles. In particular, if we denote the periods by

$$\int_{A_I} \Omega = X^I \tag{2.1}$$

$$\int_{B^J} \Omega = F_J \tag{2.2}$$

we can use the X^I as projective coordinates on the moduli space of the Calabi-Yau (in particular special geometry implies that F_J is determined in terms of X^I as gradients of the prepotential $\mathcal{F}_0$, i.e. $F_J = \partial_J \mathcal{F}_0(X^I)$). However, as observed in [2], X^I and $\overline{X}^I$ do not commute in the BPS mini-superspace. Therefore, to write the wave function we have to choose a commuting subspace. A natural such choice is to parameterize this subspace by either real or imaginary part of X^I. Let us call these variables Φ^I. Then, the wave function is given by

$$\psi_{P^I,Q_J}(\Phi^I) = \psi_{top}(P^I + \frac{i\Phi^I}{\pi}) \exp(Q_J \Phi^J/2) \tag{2.3}$$

where

$$\psi_{top}(X^I) = \exp(\mathcal{F}_{top}(X^I)) \tag{2.4}$$

is the B-model topological string partition function. For compact case there are only finite number of moduli, but for the non-compact case, which we are interested in here, there are infinitely many moduli and I runs over an infinite set. It should be understood that this

[1] If we change the boundary conditions on the fermions to be anti-periodic, then the wave function does depend on the radius of $\mathbf{S}^1$ and its norm increases as the value of the supersymmetry breaking parameter increases [6].

expression for the wave function is only an asymptotic expansion (see [7] for a discussion of non-perturbative corrections to this). The overall rescaling of the charges is identified with the inverse of topological string coupling constant and we assume it to be large, so that the string expansion is valid. The wave function is peaked at the attractor value where

$$\begin{aligned} \mathrm{Re} X^I &= P^I, \\ \mathrm{Re} F_J &= Q_J. \end{aligned} \tag{2.5}$$

We will be interested in a non-compact Calabi-Yau space, where the same formalism continues to hold (one can view it, at least formally, as a limit of a compact Calabi-Yau). In this case, we will have an infinite set of moduli. This is similar to [8,9], where the incorporation of the infinitely many moduli in the non-compact case was shown to be crucial for reproducing the conjecture of [3]. Specifically, in this paper we will be considering the conifold, $T^*\mathbf{S}^3$. In this case, we can turn on a set of fluxes which result in a round $\mathbf{S}^3$ at the attractor point, and then consider the fluctuations of the metric captured by the topological string wave function. Before we proceed to this analysis, let us review some aspects of the topological strings on the conifold and its relation to non-critical bosonic strings on a circle of self-dual radius.

3. Topological Strings on the Conifold and Non-critical c=1 String

In this section we review the B-model topological string on the conifold

$$xy - zw = \mu \tag{3.1}$$

deformed by the terms of the form $\epsilon(x, y, z, w)$. The canonical compact 3-cycle of the conifold is $\mathbf{S}^3$. If we rewrite (3.1) as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \mu \tag{3.2}$$

using appropriate change of the variables the real slice is exactly this $\mathbf{S}^3$ with the radius equal to $\sqrt{\mathrm{Re}\mu}$.

Let us recall that these deformations are in 1-to-1 correspondence with spin (j, j) representations of the $SO(4) \cong SU(2) \times SU(2)$ symmetry group [10,11,12]. Here, the variables x_i transform in the $(\frac{1}{2}, \frac{1}{2})$ representation of the $SU(2) \times SU(2)$. Thus, we can

write x_i as $x^{AA'}$ where $A, A' = 1, 2$ are the spinor indices. In these notations, infinitesimal deformations of the hypersurface (3.2) can be represented by monomials of the form

$$\epsilon(x) = t_{A_1 A_2 \ldots A_n ; A'_1 A'_2 \ldots A'_n} x^{A_1 A'_1} x^{A_2 A'_2} \ldots x^{A_n A'_n} \tag{3.3}$$

where the deformation parameters $t_{A_1 A_2 \ldots A_n ; A'_1 A'_2 \ldots A'_n}$ are completely symmetric in all A_i and all A'_i:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \mu + \epsilon(x) \tag{3.4}$$

We shall label a generic deformation of the form (3.3) by its quantum numbers:

$$t_{A_1 A_2 \ldots A_n ; A'_1 A'_2 \ldots A'_n} \rightarrow \ t_{|j,j;m,m'\rangle}. \tag{3.5}$$

where $j = n/2$ and

$$\begin{aligned} m &= \sum_{i=1}^{n} (A_i - 3/2) \\ m' &= \sum_{i=1}^{n} (A'_i - 3/2) \end{aligned} \tag{3.6}$$

As is well known [4,13,14], topological B-model partition function of the conifold Z_{top} considered as a function of the deformation parameters (3.5) can be identified with the partition function of the $c = 1$ non-critical bosonic string theory at the self-dual radius

$$Z_{\text{top}}(t) = Z_{c=1}(t) \tag{3.7}$$

The partition function (3.7) of the $c = 1$ theory is a generating functional for all correlation functions and has a natural genus expansion in the string coupling constant g_s. From the point of view of the conformal algebra the equation (3.1) describes a relation among four generators of the ground ring [10], and "cosmological constant" μ is interpreted as the conifold deformation parameter. Moreover the $SU(2) \times SU(2)$ used in classifying deformation parameters of the conifold get identified with the $SU(2) \times SU(2)$ symmetry of the conformal theory of $c = 1$ at the self-dual radius. Turning on only the momentum modes leads to deformations which depend on two of the parameters $\epsilon(x, y)$, whereas turning on the winding modes corresponds to deformations of the other two variables $\epsilon(z, w)$. Turning on all modes corresponds to an arbitrary deformation of the conifold $\epsilon(x, y, z, w)$, captured by (3.3).

The most well studied part of the amplitudes of $c=1$ involves turning on momentum modes only. This corresponds to deformation

$$xy - zw = \mu + \sum_{n>0} (t_n x^n + t_{-n} y^n) + \dots \tag{3.8}$$

Here the dots stand for the terms of higher order in t_n which are only a function of x and y. The deformations t_n, associated with momentum n states, have the $SU(2) \times SU(2)$ quantum numbers

$$t_n \qquad \longleftrightarrow \qquad |\frac{|n|}{2}, \frac{|n|}{2}; \frac{n}{2}, \frac{n}{2}\rangle \tag{3.9}$$

where n runs over all integers.

The partition function (3.7) for this subset of deformations is equal to the τ-function of the Toda hierarchy. In particular, it depends on infinite set of couplings which are sources for the *amputated* tachyon modes[2]

$$\langle \mathcal{T}_{n_1} \dots \mathcal{T}_{n_k} \rangle = \frac{\partial}{\partial t_{n_1}} \dots \frac{\partial}{\partial t_{n_k}} \mathcal{F}_{c=1}(t)|_{t=0} \tag{3.10}$$

where on the left hand side we have connected amplitudes. The conservation of momentum implies that the sum of n's in each non-vanishing amplitude should be equal to zero,

$$\langle \mathcal{T}_{n_1} \mathcal{T}_{n_2} \dots \mathcal{T}_{n_k} \rangle = 0 \quad \text{unless} \quad \sum_{i=1}^{k} n_i = 0 \tag{3.11}$$

The tachyon correlators (3.10) can be computed using the $\mathcal{W}_{1+\infty}$ recursion relations of 2D string theory [16]. For example, for genus 0 amplitudes we have

$$\begin{aligned}
\langle \mathcal{T}_n \mathcal{T}_{-n} \rangle &= -\frac{\mu^{|n|}}{g_s^2} \frac{1}{|n|} \\
\langle \mathcal{T}_n \mathcal{T}_{n_1} \mathcal{T}_{n_2} \rangle &= \frac{1}{g_s^2} \mu^{\frac{1}{2}(|n|+|n_2|+|n_3|)-1} \\
\langle \mathcal{T}_{n_1} \mathcal{T}_{n_2} \mathcal{T}_{n_3} \mathcal{T}_{n_4} \rangle &= \frac{1}{g_s^2} \mu^{\frac{1}{2}(|n_1|+|n_2|+|n_3|+|n_4|)-2} (1 - \max\{|n_i|\}) \\
\dots
\end{aligned} \tag{3.12}$$

[2] Here, $\mathcal{T}_n = \frac{\Gamma(|n|)}{\Gamma(-|n|)} T_n$ where $T_n = \int d^2\sigma e^{(2-|n|)\phi/\sqrt{2}} e^{inX/\sqrt{2}}$ is the standard tachyon vertex. (We follow the conventions in [15,16], which are slightly different from the conventions used in [17,18].) For $n \in \mathbb{Z}$, the vertex operator $\mathcal{T}_n$ is a linear combination of a "special state" and the tachyon vertex [19,15].

The genus expansion of the free energy has the form

$$\mathcal{F}_{c=1} = \sum_{g=0}^{\infty} \left(\frac{\mu}{g_s}\right)^{2-2g} \mathcal{F}_g(t) \tag{3.13}$$

This is a good expansion in the regime $\mu >> g_s$. Usually, it is convenient to absorb the string coupling constant in the definition of μ, and make a suitable redefinition of the t_n's. However, in our case it is convenient *not* to do this; the advantage is that t_n's appear in the deformed conifold equation (3.8) without any extra factors.

It is instructive to note that, in this set of conventions, all the parameters μ, g_s, and t_n are dimensionful:

$$\begin{aligned} \mu &\sim [\text{length}]^2 \\ g_s &\sim [\text{length}]^2 \\ t_n &\sim [\text{length}]^{2-|n|} \end{aligned} \tag{3.14}$$

In particular, the ratio $\left(\frac{\mu}{g_s}\right)$ is dimensionless, and t_n has the same dimension as $\mu^{1-\frac{|n|}{2}}$. Since the genus-g term in the free energy (3.13) should be independent of g_s, it follows that $\mathcal{F}_g(t)$ depends on t_n only via the combination $t_n \mu^{\frac{|n|}{2}-1}$. This is consistent with the fact that when all the t_n's are zero $\mathcal{F}_g$ is just a number

$$\mathcal{F}_g(t_n = 0) = \frac{(-1)^{g+1} B_{2g}}{2g(2g-2)} \quad , \quad g > 1 \tag{3.15}$$

In general, $\mathcal{F}_g(t)$ has the following structure [15], [18]

$$\mathcal{F}_g(t) = \sum_m P_g^m(n_i) \prod_{i=1}^{m} t_{n_i} \mu^{\frac{|n_i|}{2}-1} \tag{3.16}$$

where $P_g^m(n_i)$ is a polynomial in the momenta n_i of fixed degree depending on m and g. For example, $\deg P_g^2(n_i) = 4g - 1$ and

$$P_1^2(n) = \frac{1}{24}(|n| - 1)(n^2 - |n| - 1) \tag{3.17}$$

Also, $P_0^m(n_i)$ is a linear polynomial and for $m > 2$ is given by

$$P_0^m(n_i) = (-1)^{m-1} \frac{\mu^{m-2}}{m!} \Big(\psi_{m-2} + \frac{\max\{|n_i|\}}{2} \sum_{r=1}^{m-3} \frac{(m-2)!}{r!(m-2-r)!} \psi_{m-2-r} \psi_r \Big), \tag{3.18}$$

where $\psi_r := \left(\frac{d}{d\mu}\right)^r \log\mu$. Notice that these expressions for $P_0^3(n_i)$ and $P_0^4(n_i)$ agree with (3.12). Thus, the leading genus zero terms have the following form

$$\mathcal{F}_{c=1} = -\frac{1}{g_s^2}\sum_{n>0}\frac{1}{n}\mu^n t_n t_{-n} + \frac{1}{3!g_s^2}\sum_{n_1+n_2+n_3=0}\mu^{\frac{1}{2}(|n_1|+|n_2|+|n_3|)-1}t_{n_1}t_{n_2}t_{n_3} + \dots \quad (3.19)$$

It is easy to check that all the terms in this formula scale as μ^2. It also leads to the tachyon correlation functions (3.10) consistent with the KPZ scaling [20] (see also [17,18]):

$$\langle \mathcal{T}_{n_1}\mathcal{T}_{n_2}\dots\mathcal{T}_{n_k}\rangle_g \sim \mu^{2(1-g)-k+\frac{1}{2}\sum_{i=1}^k |n_i|} \quad (3.20)$$

4. Local Volume Form Fluctuations on $\mathbf{S}^3$

We are interested in making a toy model of quantum cosmology. In this regard we are interested in the quantum metric fluctuations of an $\mathbf{S}^3$ inside the Calabi-Yau. More precisely, we take the nine-dimensional spatial geometry as

$$M^9 = \mathbf{S}^1 \times \mathbf{S}^2 \times T^*\mathbf{S}^3$$

and study the fluctuations of the metric on $\mathbf{S}^3 \subset T^*\mathbf{S}^3$. Usually we view the Calabi-Yau scales as much smaller than the macroscopic scales $\mathbf{S}^1$ and $\mathbf{S}^2$, but nothing in the formalism of Hartle-Hawking wave function prevents us from considering larger Calabi-Yau. In particular we will be assuming that $\mathbf{S}^3$ has a very large macroscopic size, which we wish to identify with our observed universe. One may view our world, in this toy model, as for example coming from branes wrapped over this $\mathbf{S}^3$, as we will discuss later in this section. For the purposes of this section we assume we have certain fluxes turned on, such that the classically preferred geometry for $\mathbf{S}^3$ is a large, round metric, and we study what kind of fluctuations are implied away from this round metric, in the context of Hartle-Hawking wave function in string theory. We ask, for example, if the metric fluctuation spectrum implied by this wave function is scale invariant?

From the point of view of the flux compactification considered in section 2 we set all electric fluxes to zero and turn on only one magnetic flux:

$$\begin{aligned} Q_I &= 0, \\ P^{I\neq 0} &= 0, \\ P^0 &= N \end{aligned} \quad (4.1)$$

Then the value of μ is fixed by the attractor mechanism:

$$\mathrm{Re}\mu = \frac{1}{2}N g_s \quad (4.2)$$

We will fix $N \gg 1$ leading to $\mathrm{Re}\mu/g_s \gg 1$. Note that in this limit the topological string partition function has a well defined perturbative expansion.

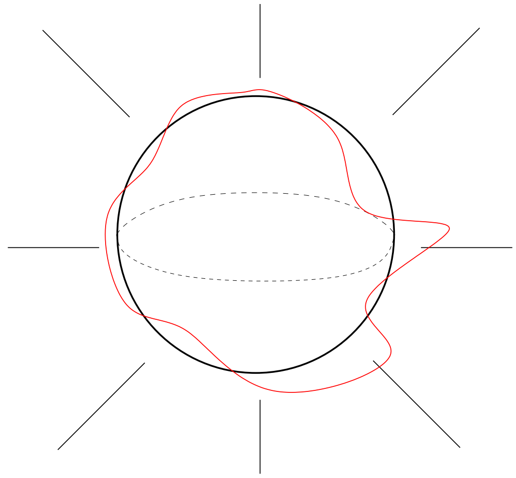


Fig. 1: Fluctuations of the $\mathbf{S}^3$ inside a Calabi-Yau.

Now, let us consider a 3-sphere, $\mathbf{S}^3$, defined by the real values of the x_i, which satisfy eq.(3.2). For non-zero (real) value of the deformation parameter μ and zero values of the t's, the induced metric on the $\mathbf{S}^3$ is the standard round metric, $g^{(0)}_{\mu\nu}$. The fluctuations of the moduli, δt, lead to perturbations of the Calabi-Yau metric on the conifold (3.2), and thus to perturbations of the metric, $g_{\mu\nu}$, induced on the 3-sphere

$$\delta t \longrightarrow \delta g_{\mu\nu} \tag{4.3}$$

In the topological B-model, the theory depends *only* on the complex structure deformations. This in particular means that not all deformations of the metric are observable. However, we recall that in the B-model the fundamental field is the holomorphic 3-form Ω and its variations. Moreover, on a special Lagrangian submanifold the volume form coincides with the restriction of a real form of Ω. In particular, an analog of the scalar fluctuations would be a fluctuation of the "conformal factor" ϕ, where

$$\Omega = e^{\phi}\Omega_0 \tag{4.4}$$

In particular the field ϕ and its fluctuations on a Lagrangian submanifold would be observable in our wave function induced from the B-model topological string. In particular we would be interested in the fluctuations of the field ϕ on the special Lagranigan $\mathbf{S}^3$ inside the conifold. Before discussing how we do this in more detail, let us return to what kind of cosmological models would this question be relevant for.

4.1. Toy Models of $\mathbf{S}^3$ *Cosmology*

So far we have discussed a supersymmetric (morally static) situation, where we ask the typical local shape of an $\mathbf{S}^3$ inside a Calabi-Yau. It is natural to ask if we can make a toy cosmology with this data, where the fluctuations we have studied would be observed as some kind of seed for inhomogeneity of fluctuations of matter.

In order to do this we need to add a few more ingredients to our story: First of all, we need to have the observed universe be identified with what is going on in an $\mathbf{S}^3$. The most obvious way to accomplish this would be in the scenario where we identify our world with some number of D3 branes wrapping $\mathbf{S}^3$. In this situation the inhomogeneities of the metric on $\mathbf{S}^3$ will be inherited by the D3 brane observer. A second ingredient we need to add to our story would be time dependence. This would necessarily mean going away from the supersymmetric context–an assumption which has been critical throughout our discussion. The least intrusive way, would be to have our discussion be applicable in an adiabatic context where we have a small supersymmetry breaking. In particular we imagine a situation where time dependence of the fields which break supersymmetry is sufficiently mild, that we can still trust a mini-superspace approximation in the supersymmetric sector of the theory.

To be concrete we propose one toy model setup where both of these can in principle be achieved. We have started with no D3 branes wrapped around $\mathbf{S}^3$. In the context of attractor mechanism this means that

$$\mathrm{Re}\widehat{\mu} = P$$

$$\mathrm{Re}\frac{i}{2\pi}\widehat{\mu}\log\frac{\widehat{\mu}}{\Lambda} = Q = 0$$

(where $\widehat{\mu} = 2\mu/g_s$) which we realize by taking $\widehat{\mu}$ to be real and equal to P and Λ to be real. The value of Λ is set by the data at infinity of the conifold. Let us write

$$\Lambda = \Lambda_0 \exp(i\varphi)$$

and imagine making Λ time dependent by taking a time dependent $\varphi(t)$. This can be viewed as a "time dependent axion field" induced from data at infinity. This leads to creation of flux corresponding to D3 brane wrapping $\mathbf{S}^3$ as is clear from the attractor mechanism. Indeed each time φ goes through 2π the number of D3 branes wrapping $\mathbf{S}^3$ increases by P units. This in turn can nucleate the corresponding D3 branes.

To bring in dynamics leading to evolution of radius of $\mathbf{S}^3$ we can imagine the following possibilities: Make the magnetic charge P time dependent by bringing in branes from infinity in the same class (or perhaps by the magnetic brane leaving and annihilating other magnetic anti-branes, leading to shrinking $\mathbf{S}^3$). This can in principle be done in an adiabatic way, thus making our story consistent with a slight time dependent μ. Another possibility which would be less under control would be to inject some energy on the D3 branes. It is likely that this leads to some interesting evolution for $\mathbf{S}^3$, though this needs to be studied. In particular, this can be accomplished by making the $\varphi(t)$ undergo partial unwinding motion. In this way we would create some number of anti-D3 branes which would annihilate some of the D3 branes. It is interesting to study what kind of cosmology this would lead to. Keeping this toy model motivation in mind we now return to the study of volume fluctuations in the supersymmetric model.

4.2. Setup for Computation of Volume Fluctuations

In principle if we know the full amplitudes of the $c = 1$ theory at the self-dual radius we can compute all correlation functions of ϕ. However the full amplitudes for $c = 1$ are not currently known (see [5] for recent work in this direction). We thus focus on the amplitudes which are known, which include the momentum mode correlations, as discussed in section 3. For two point correlation functions, as we will note below, the general amplitudes can be read off from this subspace of deformations, due to $SU(2) \times SU(2)$ symmetry of $\mathbf{S}^3$.

The momentum induced deformation of the 3-sphere in the conifold geometry (3.8) is obtained by specializing to a real three-dimensional submanifold, described by the equation

$$p + x_3^2 + x_4^2 = \mu + \epsilon(p, \theta) \tag{4.5}$$

where x_3, x_4 are real and without loss of generality, μ is assumed to be real and

$$\begin{aligned} x &= p^{1/2} e^{i\theta} \\ y &= p^{1/2} e^{-i\theta} \end{aligned} \tag{4.6}$$

In these variables, the restriction of the holomorphic 3-form Ω to the hypersurface (4.5), which is the volume form on it, is given by

$$\Omega = \frac{dx_3 dx_4 d\theta}{1 - \partial_p \epsilon(p, \theta)}, \tag{4.7}$$

In particular, in the linear approximation

$$\epsilon(p,\theta) \approx \mathrm{Re} \sum_{n\neq 0} p^{|n|/2} e^{in\theta} t_n \tag{4.8}$$

The fluctuations of the “conformal factor” are given by

$$\phi = \log \frac{\Omega}{\Omega_0} = -\log(1-\partial_p \epsilon) = \partial_p \epsilon + \tfrac{1}{2}(\partial_p \epsilon)^2 + \ldots \tag{4.9}$$

Now let us look at the absolute value squared of the Hartle-Hawking wave function (2.4). Remember that relation between the $c=1$ theory at the self-dual radius and the B-model topological string on the conifold (3.7) implies $\mathcal{F}_{top}(t) = \mathcal{F}_{c=1}(t)$. Therefore,

$$|\Psi|^2 = \exp\Big(-\frac{1}{g_s^2}\sum_{n>0}\frac{2}{n}\mu^n \mathrm{Re}(t_n t_{-n}) + \frac{1}{3g_s^2}\sum_{n_1+n_2+n_3=0} \mu^{\frac{1}{2}(n_1+n_2+n_3)-1}\mathrm{Re}(t_{n_1}t_{n_2}t_{n_3}) + \ldots\Big) \tag{4.10}$$

We are going to use this wave function density to evaluate correlation functions of the form:

$$\langle \phi_1 \phi_2 \ldots \phi_n \rangle = \frac{\int \mathcal{D}t(\phi_1\phi_2\ldots\phi_n)|\Psi(t)|^2}{\int \mathcal{D}t|\Psi(t)|^2} \tag{4.11}$$

where $\phi_k = \phi(p=\mu,\theta_k)$ is the conformal factor at a point on the “large circle” of the $\mathbf{S}^3$, defined by eq. (4.5) with $x_3 = x_4 = 0$. As we already noted, a computation of more general correlations functions (where ϕ_k are in general position on the $\mathbf{S}^3$) would require the information about the correlation functions of both momentum and winding modes of the $c=1$ model. The reason for this is that $p \neq \mu$ implies $x_3^2 + x_4^2 \neq 0$ and, therefore, leads to generic deformations $\epsilon(p,\theta,x_3,x_4)$ in (4.5). Since from now on we will always consider only the correlation functions of the conformal factor on the large circle, $p=\mu$, we shall often write $\phi(\mu,\theta) = \phi(\theta)$. We will now turn to the two point function for which the momentum correlation functions are sufficient to yield the general correlation function due to $SU(2)\times SU(2)$ symmetry.

4.3. 2-point Function at Tree Level

We start with evaluating a two point correlation function:

$$\langle \phi_1 \phi_2 \rangle$$

Because of the $SO(4)$ symmetry of the $\mathbf{S}^3$, one can always assume that ϕ_1 and ϕ_2 are evaluated at two points on the large circle $p=\mu$. To the leading order in g_s/μ, one

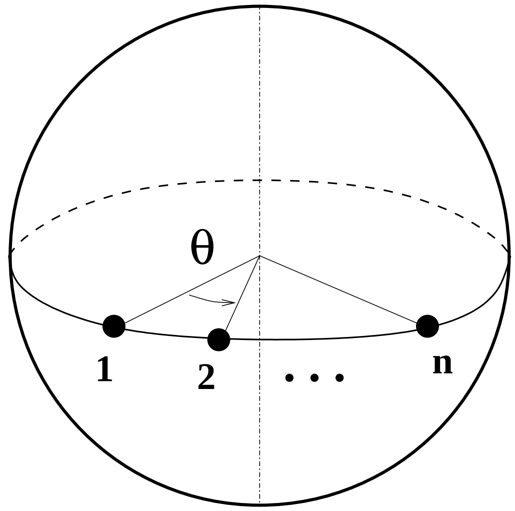


Fig. 2: n points on the large circle of the $\mathbf{S}^3$.

can keep only the linear terms in (4.9). The contribution of non-linear terms in (4.9) is suppressed by g_s and will be discussed later. Thus, in the linear approximation to (4.9) and substituting (4.8),

$$\langle\phi(\theta)\phi(0)\rangle = \frac{1}{4\mu^2}\sum_{n,m}|nm|\mu^{\frac{|n|+|m|}{2}}e^{in\theta}\langle t_n t_m\rangle \tag{4.12}$$

where we used the fact that t_n and t_{-n} are complex conjugate after reduction to the 3-sphere. Similarly, restricting (4.10) to the 3-sphere, we get

$$|\Psi_{\mathbf{S}^3}(t)|^2 = \exp\left(-\frac{2}{g_s^2}\sum_{n>0}\frac{\mu^{|n|}}{|n|}t_n t_{-n} + \frac{1}{3g_s^2\mu}\sum_{n_1+n_2=-n_3}\mu^{\frac{|n_1|+|n_2|+|n_3|}{2}}t_{n_1}t_{n_2}t_{n_3} + \ldots\right) \tag{4.13}$$

In particular, it gives

$$\langle t_n t_m\rangle = \frac{\int \mathcal{D}t|\Psi_{\mathbf{S}^3}(t)|^2 t_n t_m}{\int \mathcal{D}t|\Psi_{\mathbf{S}^3}(t)|^2} = \frac{|n|}{2}g_s^2\mu^{-|n|}\delta_{n+m,0} + \mathcal{O}((\mu/g_s)^{-|n|-2}) \tag{4.14}$$

Evaluating the path integral we treat non-quadratic terms in (4.13) as perturbations. In general, this will give a highly non-trivial theory with all types of interactions. However, using scaling properties (3.14), one can show that contribution from the k-tuple interaction vertex is proportional to $(g_s/\mu)^{k-2}$ (see discussion below). Therefore, all loop corrections to the leading term are suppressed in the limit of large $\mathbf{S}^3$ radius and small string coupling, g_s. As a result, using (4.14) we find

$$\langle\phi(\theta)\phi(0)\rangle = \frac{g_s^2}{8\mu^2}\sum_n |n|^3 e^{in\theta} + \ldots \tag{4.15}$$

or, equivalently,

$$\langle \phi_n \phi_{-n} \rangle = \frac{g_s^2}{8\mu^2} |n|^3 \tag{4.16}$$

where

$$\phi_n := \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i\theta n} \phi(\theta) \tag{4.17}$$

This has to be compared with the usual Fourier transform, given by a 3-dimensional integral

$$\begin{aligned} \langle \phi_{\vec{k}} \phi_{-\vec{k}} \rangle &\sim \int d^3 x e^{i\vec{k}\cdot\vec{x}} \langle \phi(\vec{x}) \phi(0) \rangle \\ &\sim g_s^2 |\vec{k}| \end{aligned} \tag{4.18}$$

Note that a scale invariant power spectrum would correspond to $|k|^{-3}$ fluctuation correlation. Thus the fluctuation spectrum we have on $\mathbf{S}^3$ is *not* scale invariant.

After performing the summation over n in (4.16), we get the 2-point function

$$\langle \phi(\theta) \phi(0) \rangle = \frac{g_s^2}{32\mu^2} \frac{\cos\theta + 2}{\sin^4 \theta/2} \tag{4.19}$$

in the coordinate representation. Although this expression appears to have a singularity at $\theta = 0$, as we explain in the next section, our approximation cannot be trusted at large momenta or, equivalently, small $\theta < \sqrt{g_s/\mu}$.

4.4. General Structure of g_s Corrections

There are three sources for the g_s corrections to the 2-point function: i) one due to higher genus terms in the free energy expansion (3.13), ii) corrections due to loops made from the k-point vertices (with $k > 2$) in the "effective action" $\mathcal{F}(t)$ as well as due to non-linear terms in the expansion (4.9) of ϕ in terms of ϵ, and iii) corrections due to non-linear relation between ϵ and the deformation parameters t_n induced by the deformations of the geometry [16,21]:

$$xy = \mu - x_3^2 - x_4^2 + \sum_{n>0} (t_n x^n + t_{-n} y^n) - \frac{1}{2\mu} \sum_{\substack{m>0 \\ n>0}} t_n t_{-m} m x^n y^m + \ldots \tag{4.20}$$

It is easy to check that all kinds of corrections are suppressed by powers of $(g_s/\mu)^2$. In the case i) this is manifest from the form of (3.13). In the case $ii), iii$), this can be seen in

the language of the Feynman diagrams for the fluctuating fields t_n, that follow from the effective action (4.13):

$$\begin{array}{ll} \text{propagator :} & \frac{1}{2} g_s^2 |n| \mu^{-|n|} + \dots \\ k \geq 3 \text{ vertex :} & \frac{2}{g_s^2 k!} \mu^{\frac{|n_1|+\dots+|n_k|}{2}+2-k} P_0^k(n_i) + \dots \end{array} \tag{4.21}$$

Here $P_0^k(n_i)$ is a linear polynomial (3.18) in momenta n_i, and the dots stand for higher-genus terms. In particular, a genus-g contribution comes with an extra factor of $(g_s/\mu)^{2g}$.

In general, we find that the genus-g contribution (contribution from g loops) to the 2-point function looks like

$$\langle \phi_n \phi_{-n} \rangle_g \sim \left(\frac{g_s}{\mu} \right)^{2g+2} |n|^{4g+3} \tag{4.22}$$

where ϕ_n is defined in (4.17). We can read off the higher genus corrections to the propagator $\langle t_n t_{-n} \rangle$ from the quadratic terms in (3.16):

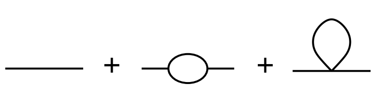
$$\langle t_n t_{-n} \rangle \big|_{tree} = \frac{\frac{1}{2} g_s^2 |n| \mu^{-|n|}}{1 + \sum\limits_{g \geq 1} (g_s/\mu)^{2g} |n| P_g^2(n)} \tag{4.23}$$

Notice that $\deg P_g^2(n) = 4g - 1$ and therefore for large momenta we have an asymptotic expansion of the form:

$$\langle t_n t_{-n} \rangle \big|_{tree} = \frac{\frac{1}{2} g_s^2 |n| \mu^{-|n|}}{1 + \sum\limits_{g \geq 1} p_g \left(\frac{n^2 g_s}{\mu} \right)^{2g} + \dots} \tag{4.24}$$

where constants p_g are determined by the polynomial $P_g^2(n)$ and dots stand for the terms suppressed by powers of n. Now it is clear that the good expansion parameter is $\frac{n^2 g_s}{\mu}$ rather than $\frac{g_s}{\mu}$ which means that our approximation is valid only for momenta n small compared to μ/g_s. In other words, we should fix some high-energy cut-off parameter $\Lambda^2 < \mu/g_s$ and consider only deformations with momentum number $n < \Lambda$.

Now let us incorporate corrections due to loops in Feynman diagrams generated by (4.21). For example, if we take into account genus one corrections and one-loop corrections we get the following expression for the propagator:



$$+ \dots = \frac{\frac{1}{2} g_s^2 |n| \mu^{-|n|} + \dots}{1 - \frac{|n|}{24} \left(\frac{g_s}{\mu} \right)^2 (|n| - 1)(n^2 - |n| - 1) + \dots} \tag{4.25}$$

Notice that due to (4.22) g-loops corrections dependence on momenta is similar to the genus g corrections. Thus, the general structure of the 2-point function is given by:

$$\langle \phi_n \phi_{-n} \rangle_g = \frac{g_s^2}{8\mu^2} |n|^3 \frac{1 + \sum_g b_g\left(\frac{n}{\Lambda}\right)\left(\frac{\Lambda^2 g_s}{\mu}\right)^{2g} + \dots}{1 + \sum_g p_g \left(\frac{n^2 g_s}{\mu}\right)^{2g} + \dots} \tag{4.26}$$

where the polynomials b_g depend on the ratio n/Λ, which should be small in order for the perturbation theory to be valid.

4.5. n-point Function for the Perturbations on the Large Circle of $\mathbf{S}^3$

Here we briefly discuss the structure of n point function. Unlike the 2-point function where we could compute the general case, for n point functions with the present technology, we can only compute correlations restricted to taking the fluctuations at points on a large circle. Using the Feynman rules (4.21), we find that the contribution of a tree Feynman diagram to a k-point function scales as (to avoid cluttering, we omit polynomials in n_i which do not affect the g_s behavior):

$$\langle t_{n_1} \dots t_{n_k} \rangle_0 \sim g_s^{2k-2} \mu^{-\frac{|n_1|+\dots+|n_k|}{2}+2-k} \tag{4.27}$$

Now, let us consider a g-loop contribution to the k-point function. As we discussed earlier, such contributions come from the vertices with a total of $k+2g$ legs, k of which are connected by propagators to the external legs of the k-point function, and $2g$ of which are pairwise connected by internal propagators. Notice that, for the internal momenta n_j, the factors $\mu^{\frac{|n_j|}{2}}$ cancel out and we get

$$\langle t_{n_1} \dots t_{n_k} \rangle_g \sim g_s^{2k+2g-2} \mu^{-\frac{|n_1|+\dots+|n_k|}{2}+2-k-2g} \tag{4.28}$$

Comparing this expression with (4.27), we see that a g-loop contribution to the k-point correlation function is suppressed by the same factor $(g_s/\mu)^{2g}$ as the contribution from a genus-g term in the free energy (3.13). For the k-point function of the fields ϕ_n this implies

$$\begin{aligned} \langle \phi_{n_1} \dots \phi_{n_k} \rangle_g &\sim \mu^{\frac{|n_1|+\dots+|n_k|}{2}-k} \langle t_{n_1} \dots t_{n_k} \rangle_g \\ &\sim \left(\frac{g_s}{\mu}\right)^{2k+2g-2} \end{aligned} \tag{4.29}$$

where in the second line we used (4.28). Notice, this structure is consistent with our results (4.16) for the 2-point function. For an example of higher point function we now turn to a discussion of the leading correction to the 3-point function.

4.6. 3-point Function

Now, let us look more carefully at the structure of the 3-point function. Unlike the 2-point function where we studied the general case, since the topological string amplitudes are not known for arbitrary deformations of the conifold, we restrict our attention to the ones corresponding to momentum modes. This means that we consider 3-point functions where all three points lie on the large circle of the $\mathbf{S}^3$. To the leading order in $(g_s/\mu)^2$, from (4.9) we find

$$\langle \phi(\theta_1)\phi(\theta_2)\phi(\theta_3) \rangle = \frac{1}{8\mu^3} \sum_{n,m,l} |nml| \mu^{\frac{|n|+|m|+|l|}{2}} e^{in\theta_1 + im\theta_2 + il\theta_3} \langle t_n t_m t_l \rangle \tag{4.30}$$

In the momentum representation, this looks like

$$\langle \phi_n \phi_m \phi_l \rangle = \frac{1}{8\mu^3} |nml| \mu^{\frac{|n|+|m|+|l|}{2}} \langle t_n t_m t_l \rangle \tag{4.31}$$

According to (4.13),

$$\langle t_n t_m t_l \rangle = \frac{g_s^4}{4} \delta_{n+m+l,0} |nml| \mu^{-\frac{|n|+|m|+|l|}{2} - 1} \tag{4.32}$$

which gives

$$\langle \phi_n \phi_m \phi_l \rangle = \frac{1}{32} \left(\frac{g_s}{\mu} \right)^4 |nml|^2 \delta_{n+m+l,0} \tag{4.33}$$

This is to be compared with the three point function of fluctuations in the inflationary cosmology [22].

5. Further Directions

In section 4 we mentioned some potential cosmological models based on the $\mathbf{S}^3$ in the conifold geometry. It would be very interesting to develop these ideas further. It is also natural to ask if in some other non-compact models we could get a different spectrum of fluctuations. It is probably true that the local Calabi-Yau fluctuations that we find are fairly generic, but the global aspects might be different for different Calabi-Yau.

As for the local fluctuations, it would be interesting to relate our results to properties of the Kodaira-Spencer theory, which is the target space theory of topological gravity. In particular, it should be possible to derive the 2-point function of the local fluctuations directly from the relation between the Kodaira-Spencer theory, the B-model, and its interpretation as the Hartle-Hawking wave function.

Another aspect of the wave function that topological strings lead to has to do with the fundamental question of 'which' Calabi-Yau space is preferred? Naively, one would think that it is not possible to answer this question in our setup, because the probability density depends on the scale of the holomorphic 3-form, which in turn corresponds to overall rescaling of the black hole charge. In particular, the overall rescaling of the black hole charge does not affect the attractor point on the moduli space of the Calabi-Yau, but makes the entropy of the black hole arbitrarily large for *any* point on the moduli of Calabi-Yau. However, it turns out that there is a way to extract information about which Calabi-Yau is preferred [23]: We consider a local model of a Calabi-Yau, very much in line with the example considered in this paper. In this way we can partially decouple gravity from our wave function by fixing the period of the holomorphic 3-form on a particular 3-cycle,

$$\int_{A_0} \Omega \sim \frac{1}{g_s}$$

With this constraint, we then look for the local maxima of the entropy functional,

$$S = -i\frac{\pi}{4} \int_{CY} \Omega \wedge \overline{\Omega}.$$

It turns out, that certain singularities of the Calabi-Yau on the moduli space extremize S subject to the above constraint. Moreover, in these models, we find a correlation between the sign of the beta function in the effective four-dimensional field theory and the second order variation of the functional S. In particular, we find that S is maximized *for asymptotically free theories*! Further details of these models will appear elsewhere [23].

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